

Fredholm Integral Equations Of The First Kind

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معادلات فريدهولم التكاملية من النوع الأول

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Abstract:

In this paper, focus on the exploration of Fredholm integral equations of the first kind. We're set to tackle the challenges of illconditioned Fredholm problems by using the regularization method alongside current strategies. We'll showcase the method's effectiveness through a range of examples.

Keywords: integral equations , Fredholm Integral Equations , Volterra integral equation , Regularization method

الملخص

يتناول هذا البحث دراسة معادلات فريدهولم التكاملية من النوع الأول. ونسعى من خلاله إلى معالجة التحديات المرتبطة بمسائل فريدهولم "سيئة الاعتلال" (ill-conditioned problems) باستخدام طريقة التنظيم (regularization method) إلى جانب استراتيجيات أخرى معاصرة. كما نوضح مدى فعالية هذه الطريقة من خلال عرض مجموعة من الأمثلة.

الكلمات المفتاحية: المعادلات التكاملية، معادلات فريدهولم التكاملية، معادلات فولتيرا التكاملية، طريقة التنظيم.

1. Introduction

Back in 1825, Italian mathematician Abel made a significant impact in this area by crafting a n integral equation tied to the famous tautochron problem. Integral equations, which involve evaluating an unknown function within the integral sign, are crucial in both pure and applied mathematics. This field of study has provided numerous solutions to initial and boundary value problems related to ordinary differential equations (ODE) and partial differential equations (PDE), while also offering the means to reformulate them.

Integral equations are a type of equation where the unknown function is part of the integral. They can be classified based on certain properties from Linearity: If the unknown function and its integrals appear to be linear, the equation is linear. If not, the equation is nonlinear. If the right-hand side of the equation is equal to zero, the equation is homogeneous If the limits of integration are constant, it is a Fredholm integral equation. If one of the limits is a variable, it is a Volterra integral equation (Guo, Gustavson, and Li 2020) Integral equations play a crucial role in both pure and applied mathematics, providing solutions

to many initial and boundary value problems related to ordinary differential equations (ODEs) and partial differential equations (PDEs). Singular and regular: This classification is based on the behavior of the kernel of the integral equation. Type first and Type second If the unknown function lies under the integral sign only, the equation is said to be of “Type first”. If the function is both internal and external, it is called an equation of the second type. (Masouri and Hatamzadeh 2020; Ogola n.d.)

Fredholm integral equations are a class of integral equations that have been extensively studied in the field of mathematics. These equations are named after the Swedish mathematician Ivar Fredholm, whose pioneering work significantly advanced their understanding. A Fredholm integral equation is typically expressed as an operator equation, involving the Fredholm integral operator denoted by k . This operator acts on functions, transforming them according to the following definition:

$$u(x) = f(x) + \lambda \int_a^b k(x, y) u(y) dy \quad (1)$$

In the context of Fredholm integral equations, a and b are constants, while λ is a parameter. The function $f(x)$ is known as the data function, and $k(x, y)$ is the kernel of the integral equation. The function $u(x)$ is the unknown function that we aim to determine. Equation (1) is referred to as the Fredholm integral equations of the second kind. This is due to the presence of the unknown function $u(x)$ both inside and outside the integral sign. [2,3,]

On the other hand, Fredholm integral equations of the first kind are characterized by a different form.

$$f(x) = \lambda \int_a^b k(x, y) u(y) dy \quad (2)$$

Fredholm integral equations of the first kind (2) are characterized by the occurrence of the unknown function $u(x)$ only inside the integral sign. The existence of $u(x)$ inside the integral sign causes special difficulties. Fredholm integral equations of the first kind are often ill-posed problems, that may have no solution, or if a solution exists it is not unique and may not depend continuously on the data $f(x)$. The Fredholm integral equations of the first kind (2) appear in many physical models

2. The regularization method

The regularization method is the most common approach for solving the first kind of Fredholm integral equation. Tikhonov's regularization method is particularly effective for obtaining stable solutions to the original problem. In recent years, many researchers have tackled the first kind of Fredholm integral equations with various regularization techniques, drawing significant conclusions. Lee and Prenter addressed the numerical solution of the first kind of Fredholm integral equation, while Perminov applied Tikhonov's method for the numerical solution of the same. The integral equation is transformed from the first kind to the second kind through independent regularization.

$$f(x) = \int_a^b k(x, y) u(y) dy \quad (3)$$

The previous Fredholm integral equation approximates

$$A u(x) = f(x) - \int_a^b k(x,y) u(y) dy \quad (4)$$

where A is the regularization parameter.

We conclude that the previous equation is an equation of the second type and can be written in the following form

$$u(x) = \frac{1}{A} f(x) - \frac{1}{A} \int_a^b k(x,y) u(y) dy \quad (5)$$

The last equation can be solved converges when $A \rightarrow 0$. it was shown that

$$v(x) = \lim_{A \rightarrow 0} u(x) \quad (6)$$

3. Fredholm integral equation of the first kind

we will employ the regularization method in conjunction with a suitable technique. It's important to note that the data function, $f(x)$, must comprise components that align with the corresponding x components of the kernel, $K(x,y)$. This alignment is a prerequisite for ensuring a solution

Example solve the Fredholm integral equation of the first kind

$$2x = \int_0^1 x u(t) dt$$

$$A u(x) = 2x - \int_0^1 x u(t) dt$$

$$u(x) = \frac{2}{A} x - \frac{1}{A} \int_0^1 x u(t) dt$$

To use the successive approximations method,. Consequently, we obtain the following approximations

Example solve the Fredholm integral equation of the first kind

$$\frac{1}{2} \sin(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin(x) \sin(t) u_\alpha(t) dt, \quad 0 < x < \frac{\pi}{2}$$

$$\alpha u_\alpha(x) = \frac{\sin(x)}{2} - \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin(x) \sin(t) u_\alpha(t) dt$$

$$u_{\alpha}(x) = \frac{\sin(x)}{2\alpha} - \frac{2}{\pi\alpha} \int_0^{\frac{\pi}{2}} \sin(x) \sin(t) u_{\alpha}(t) dt$$

$$u_0(x) = \frac{\sin(x)}{2\alpha}$$

$$u_{\alpha+1}(x) = -\frac{2}{\pi\alpha} \int_0^{\frac{\pi}{2}} \sin(x) \sin(t) u_{\alpha}(t) dt$$

$$u_{\alpha 1}(x) = -\frac{2}{\pi\alpha} \int_0^{\frac{\pi}{2}} \sin(x) \sin(t) u_0(t) dt$$

$$u_{\alpha 1}(x) = \frac{-2\sin(x)}{\pi\alpha} \int_0^{\frac{\pi}{2}} \sin(t) \cdot \frac{\sin(t)}{2\alpha} dt$$

$$u_{\alpha 1}(x) = \frac{-\sin(x)}{\pi\alpha^2} \int_0^{\frac{\pi}{2}} \sin(t) \cdot \sin(t) dt$$

$$u_{\alpha 1}(x) = \frac{-\sin(x)}{\pi\alpha^2} \int_0^{\frac{\pi}{2}} \sin^2(t) dt$$

$$u_{\alpha 1}(x) = \frac{-\sin(x)}{\pi\alpha^2} \cdot \frac{\pi}{4} = \frac{-\sin(x)}{4\alpha^2}$$

$$u_{\alpha 2}(x) = -\frac{2}{\pi\alpha} \int_0^{\frac{\pi}{2}} \sin(x) \sin(t) u_{\alpha 1}(t) dt$$

$$u_{\alpha 2}(x) = -\frac{2\sin(x)}{\pi\alpha} \int_0^{\frac{\pi}{2}} \sin(t) \frac{-\sin(t)}{4\alpha^2} dt$$

$$u_{\alpha 2}(x) = \frac{2\sin(x)}{4\pi\alpha^3} \int_0^{\frac{\pi}{2}} \sin^2(t) dt$$

$$u_{\alpha 2}(x) = \frac{\sin(x)}{2\pi\alpha^3} \int_0^{\frac{\pi}{2}} \sin^2(t) dt$$

$$u_{\alpha 2}(x) = \frac{\sin(x)}{2\pi\alpha^3} \int_0^{\frac{\pi}{2}} \sin^2(t) dt$$

$$u_{\alpha 2}(x) = \frac{\sin(x)}{2\pi\alpha^3} \cdot \frac{\pi}{4} = \frac{\sin(x)}{8\alpha^3}$$

$$u_{\alpha 3}(x) = -\frac{2}{\pi \alpha} \int_0^{\frac{\pi}{2}} \sin(x) \sin(t) u_{\alpha 2}(t) dt$$

$$u_{\alpha 3}(x) = -\frac{2 \sin(x)}{\pi \alpha} \int_0^{\frac{\pi}{2}} \sin(t) \frac{\sin(t)}{8 \alpha^3} dt$$

$$u_{\alpha 3}(x) = -\frac{2 \sin(x)}{\pi \alpha} \int_0^{\frac{\pi}{2}} \sin(t) \frac{\sin(t)}{8 \alpha^3} dt$$

$$u_{\alpha 3}(x) = -\frac{2 \sin(x)}{8 \pi \alpha^4} \int_0^{\frac{\pi}{2}} \sin^2(t) dt$$

$$u_{\alpha 3}(x) = -\frac{\sin(x)}{4 \pi \alpha^4} \cdot \frac{\pi}{4} = \frac{-\sin(x)}{16 \alpha^4}$$

In general, the equation is as follows:

$$u_{\alpha}(x) = \frac{\sin(x)}{2 \alpha} - \frac{\sin(x)}{4 \alpha^2} + \frac{\sin(x)}{8 \alpha^3} - \frac{\sin(x)}{16 \alpha^4} + \dots$$

$$u_{\alpha}(x) = \frac{\sin(x)}{2 \alpha} \left(1 - \frac{1}{4 \alpha^2} + \frac{1}{8 \alpha^3} - \frac{1}{16 \alpha^4} + \dots \right)$$

This result gives

$$u_{\alpha}(x) = \frac{1}{2 \alpha} \sin(x) \left(1 - \frac{1}{2 \alpha} + \frac{1}{4 \alpha^2} - \frac{1}{8 \alpha^3} + \dots \right)$$

Where

$$1 - \frac{1}{2 \alpha} + \frac{1}{4 \alpha^2} - \frac{1}{8 \alpha^3} + \dots = \frac{2 \alpha}{1}$$

Where

$$u(x) = \lim_{\alpha \rightarrow 0} u_{\alpha}(x) = \sin(x)$$

$$\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin(x) \sin(t) \phi(t) dt = \frac{\sin(x)}{2}$$

$$\sin(x) \int_0^{\frac{\pi}{2}} \sin(t) \phi(t) dt = \frac{\pi \sin(x)}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin(t) \phi(t) dt = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin(t) \frac{\pi}{4} dt = \frac{-\pi}{4} \cos(t) \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

4 .Volterra integral equation of the first kind

The Volterra integral equations are named after the Italian mathematician Vito Volterra who introduced them¹. These equations are a special type of integral equations and are divided into two groups referred to as the first and the second kind

They were further studied by Traian Lalescu in his 1908 thesis, “Sur les equations de Volterra,” written under the direction of Émile Picard

A linear Volterra equation of the first kind is represented as:

$$\int_a^x K(x, t) \phi(t) dt = f(x)$$

Notably, Volterra integral equations of the first kind are not classified as illposed problems. These equations can be converted into second kind equations by differentiating both sides with respect to x and applying Leibnitz's rule, provided the kernel K(x,t) is nonsingular. Interestingly, the regularization method is rarely used for solving Volterra integral equations of the first kind. More practical approaches, such as the Laplace transform method and differentiation, are generally favored over the regularization method. Nevertheless, for the sake of comparison, the following example will be solved using both the regularization method and the Laplace transform method (Burova and Ryabov 2021; Guo, Gustavson, and Li 2020)

.Example Solve the Fredholm integral equation of the first kind

$$x e^x = \int_0^x e^{x-t} u(t) dt$$

$$\alpha u_\alpha(x) = x e^x - \int_0^x e^{x-t} u_\alpha(t) dt$$

$$u_\alpha(x) = \frac{x e^x}{\alpha} - \frac{1}{\alpha} \int_0^x e^{x-t} u_\alpha(t) dt$$

$$u_\alpha(x) = \sum_{n=0}^{\infty} u_{\alpha n}(x)$$

$$u_{\alpha 0} = \frac{1}{\alpha} x e^x$$

$$u_{\alpha 1} = \frac{1}{2! \alpha^2} x^2 e^x$$

$$u_{\alpha 2} = \frac{1}{3! \alpha^3} x^3 e^x$$

$$u_{\alpha 3} = \frac{1}{4! \alpha^4} x^4 e^x$$

$$u_{\alpha 4} = \frac{1}{5! \alpha^5} x^5 e^x$$

Substituting into the equation

$$u_{\alpha}(x) = \sum_{n=0}^{\infty} u_{\alpha n}(x)$$

we get

$$u_{\alpha}(x) = e^x \left(\frac{x}{\alpha} - \frac{x^2}{2! \alpha^2} + \frac{x^3}{3! \alpha^3} - \frac{x^4}{4! \alpha^4} + \dots \right)$$

Then

$$u_{\alpha}(x) = e^x \left(1 - e^{-\frac{x}{\alpha}} \right)$$

$$u_{\alpha}(x) = \lim_{\alpha \rightarrow 0} u_{\alpha}(x) = e^x$$

Using the Laplace transform method

$$\frac{1}{(s-1)^2} = \frac{1}{s-1} U(s)$$

$$U(x) = \frac{1}{s-1}$$

$$U(x) = \mathcal{L}\{u(x)\}$$

$$u(x) = e^x$$

Conclusion

In this work we employed the regularization method combined with some of the proper well-known techniques to handle the Fredholm integral equations of the first kind. The method showed reliability in handling these ill-posed problems. Although the regularization method is not commonly used for the first kind of Volterra integral equations, we used the method in this case for comparison with the Laplace transform method.

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