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Comparative Analytical study of Fractional Differential Equations via Abaaub - Shkheam, Mellim, and Laplace Transforms

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"دراسة تحليلية مقارنة للمعادلات التفاضلية الكسرية باستخدام تحويلات عبعوب-شخيم، ميلين، و تحويلات لابلاس"

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Abstract:
The Paper presents a rigorous comparator analysis of fractional differential equation $F(t)$ of order $\alpha = 1/2$. We investigate the analytical solutions obtained through the modern Abaoub - Shkheam Transform, the complex - domain Mellin Transform, and the classical Laplace Transform. The study demonstrates that while the methodologies differ in their operational domains, they converge to a unique solution involving the Mittay-leffler function. The efficiency and parametric flexibility of the Abaoub-Shkheam Transform are highlighted.

Keywords: Fractional Differential Equations, Comparative Analysis, Laplace Transform, Mellin Transform Abaoub–Shkheam Transform, Analytical Methods, Integral Transforms Fractional Calculus, Exact Solutions.

الملخص
تقدم هذه الورقة دراسة تحليلية مقارنة دقيقة لمعادلة تفاضلية كسرية ألفا يساوي $\alpha = 1/2$ من الرتبة. نقوم بدراسة الحلول التحليلية التي يتم الحصول عليها باستخدام تحويل عبعوب – شخيم الحديث، وتحويل ميلين في المجال العقدي، بالإضافة إلى تحويلات لابلاس.

تُظهر الدراسة أنه على الرغم من اختلاف هذه المناهج في مجالاتها التشغيلية، إلا أنها تتقارب نحو حل وحيد يتضمن دالة ميتاغ – ليفلر. كما تُبرز النتائج كفاءة تحويل عبعوب – شخيم ومرونته البارامترية في معالجة هذا النوع من المعادلات.

الكلمات المفتاحية: المعادلات التفاضلية الكسرية، التحليل المقارن، تحويل لابلاس، تحويل ميلين، تحويل أباووب-شكهام، الطرق التحليلية، التحويلات التكاملية، حساب التفاضل والتكامل الكسري، الحلول الدقيقة.

Introduction

Fractional calculus has emerged framework as a superior for modeling non-local phenomena and memory-dependent systems in applied.

The challenge lies in the transition from the time domain to a transform domain where fractional operators become algebraically manageable. We consider the following Riemann – Liowille fractional differential equation

$$ {}_0D_t^{\frac{1}{2}} \; f(t) \; + \; a\, f(t) \; = 0, \quad t > 0 $$

1. Laplace Trans forms of Fractional Derivatives.

Let us recall some basic facts about the Laplace transform. The function $\mathbf{F}(s)$ of the complex variable $\mathbf{S}$ defined by

$$ f(t) = L\{f(t) : s\} = \int_0^{\infty} e^{-st} \; f(t) dt \; . \quad \to ① $$

Is called the Laplace transform of the function $f(t)$. The function $f(t)$ must be of exponential order $\propto$. which means that there exist Positive constants **M** and **T**

Such that:

$$ e^{-\propto t} \, |f(t)| \le M \;\; \text{for all} \quad t > \tau $$

The original $f(t)$ can be restored form the Laplace transform $f(s)$ with the help of the inverse Laplace transform

$$ f(t) = L^{-1} \{f(t); t\} = \int_{c-i\infty}^{c+i\infty} e^{-st} \; f(t) \; ds . c = Re(s) > c_0 \;\; \to ② $$

the Laplace transform of the convolution:

$$ f(t) * g(t) = \int_0^t f(t-\tau) \;\; g\,(\tau) \; d\tau = \int_0^t f(\tau) g(t-\tau) dt \;\; \to ③ $$

of the two functions $f(t)$ and $g(t)$ for $t < 0$ is equal to the Product of the laplace transform of these function

$$ L\{f(t) * g(t); s\} \; = \; f(s). \; G(s) \qquad \to ④ $$

Another useful property which we need is the formula for the Laplace transform of the derivative an integer order n of the function $f(t)$

$$ L\{f^n \,(t); S\} = \; s^n \;\; f(s) - \textstyle\sum_{k=0}^{n-1} s^{n-k-1} \, f^{(k)} \, (0) \; = \; s^n \, f(s) - \textstyle\sum_{k=0}^{n-1} s^k \;\; f^{(n-k-1)}(0) \;\; \to ⑤ $$

Laplace transform of the Rismann - Liouville fractional derivative of order $P > \; 0$

$$ L\left\{ {}_0D_t^p f(t); s \right\} = \; s^p f(s) - \; \textstyle\sum_{k=0}^{n-1} s^k \left[\, {}_0D_t^{p-k-1} \right]_{t=0} \;\; n-1 \le P < n \quad \to ⑥ $$

Laplace transform formula the Caputo Fraction derivative:

$$ L\left\{ {}_0^cD_t^p \; f(t) \right\} = \; s^p f(s) - \sum_{k=0}^{n-1} s^{p-k-1} \;\; f^{(k)}(0) $$

$$ n - 1 < P \le n \quad \to ⑦ $$

we can establish the following formula for the Laplace transform of the sequential derivative

$$ L\{{}_0D_t^{6m} \, f(t); S\} = \; 6_s^m \;\; f(t) \sum_{k=0}^{m-1} s^{6m-6m-k} \quad \left[{}_0D_t^{6m-k-1} \quad f(t) \right]_{t=0} \quad \to ⑧ $$

The Laplace transform formula for Riemann e Liouville Practional in the case $o < \alpha \le 1$ takes the form:

$$ L\,\{{}_0D_t^{\alpha} \; f(t); s\} = \; S^{\alpha} \, f(s) - [{}_0D_t^{\alpha-1} \;\; f(t)]_{t=0} \quad \to ⑨ $$

and then use the formula (9) subsequently **m** times:

$$ L\,\{{}_0D_t^{6m} \, f(t); s\} = \; L \;\; \{{}_0D_t^{\alpha m} \;\; {}_0D_t^{m-1} f(t); s\,\} \; = \; S^{6m} \;\; f(t) \sum_{k=0}^{m-1} 8^{6m-6m-k} \left[{}_0D_t^{6m-k-1} \, f(t) \, \right]_{t=0} \quad \to ⑩ $$

2. Mellin Trans forms of Fractional Derivatives:

$$ f(s) = \; M\{\, f(t); s\} = \int_0^{\infty} f(t) \; t^{s-1} \;\; dt \quad \to ① $$

- where s is complex such as:

$$r_1 < Re(s) < r_1$$

- It follows from the definition **(1)** that

$$M\{t^{\alpha} f(t);\ s\} = M\{f(t); s+\alpha\} = f(s{+}\propto). \rightarrow \text{②}$$

- The Mellin transform of the mellin convolution

$$f(t) * g(t) = \int_0^{\infty} (t\tau)\ g(\tau)\, d\tau \qquad \rightarrow \text{③}$$

Of functions $f(t)$ and $g(t)$ the mellin transforms of which $F(t)$and $g(s)$ is given by the formula:

$$M\left\{ \int_0^{\alpha} f(t\,\tau) g(\tau)\, d\tau; s \right\} = f(s)\, G(1-s) \quad \rightarrow \text{④}$$

- Mellin Transform of the capt to Fractional Derivative:

$$M\,\{{}_0^{c}D_t^{\propto} f(t)\,\} = \frac{\lceil(1-s+\alpha)}{\lceil(1-s)} f(s{-}\propto) \quad \rightarrow \text{⑤}$$

3. Abaob - Shkheam Transform Q - Transform

$$T(u,s) = Q[f] = \int_0^{\infty} f(t)\, e^{-\frac{1}{2}}\ dt \rightarrow \text{①}$$

$f(t)$ Function defined for all $t \geq 0$ the Q - Transform of $f(t)$ is the $T(u,s)$. Provided the integral exists for some S. where $SG(-t_{1,}t_2)$
The original function $f(t)$ in (1) is called the inverse of $T(u,s)$, and is denoted by

$$f(t) = Q^{-1}\{T(u,s)\}$$

If we substitute $ut = y$, then eq (1) becomes:

$$Q\,[f(t)] = T(u,s) = \frac{1}{4} \int_0^{\infty} f(y) e^{-\frac{y}{vs}} dy \qquad \rightarrow \text{②}$$

Laplace - Q duality property:

if the Laplace transform of the function is $f(t)$ is $f(s)$

$$f(s) = L\left(f(t)\right) = \int_0^{\infty} f(t)\, e^{-st}\ dt \qquad \rightarrow \text{③}$$

Substitule $t = uy$ in the integral on right hand

Proof : $T(u,s) = \frac{1}{4} F\left(\frac{1}{us}\right).$

$$F(s) = L\{f(t)\} = u \int_0^{\infty} f(u\,y)\, e^{-suy}\, \mathrm{dy}$$

$$F(s) = uT\left(u, \frac{1}{us}\right) = T(u,s) = \frac{1}{4} F\left(\frac{1}{us}\right)$$

Applications

$${}_0D_t^{\frac{1}{2}}\ f(t) + a\, f(s) = 0. \quad t > 0$$

subject to the fractional initial condition

$$\left[{}_0D_t^{-\frac{1}{2}}\ f(t)\right]_{t=0} = c$$

Methodological Frameworks:

4.1. The Abacub - shkheam Transform Method.

The Abacub-shkheam transform or Q - transform, of afunction $f(t)$ is defined by

$$Q[f(t)] = A(v,s) = \frac{1}{v} \int_0^{\infty} f(t)\, e^{-\frac{t}{vs}}\, \mathrm{dt} \qquad \rightarrow \text{①}$$

Applying the fractional derivative property

$$\mathrm{Q}[{}_0\mathrm{D}_\mathrm{t}^{\propto}\ \mathrm{f(t)}] = \frac{1}{(\mathrm{vs})^{\propto}}\ \mathrm{A(v,s)} - \frac{1}{\mathrm{v(us)}^{\propto-1}} [{}_0\mathrm{D}_\mathrm{t}^{\propto-1}\ \mathrm{f(t)}]_{\mathrm{t=0}} \rightarrow \text{②}$$

Step1: Applying the transform apply the Abaob -Shkheam transform to both sides of equation:

$$Q\left[{}_0D_t^{-\frac{1}{2}}\; f(t)\right] + a\,Q[f(t)] = 0 \quad \rightarrow \text{③}$$

Step2: Substitution of the fractional derivative. Property substituted $\alpha = \frac{1}{2}$ into the derivative property formula

$$\left(\frac{1}{(vs)^{\frac{1}{2}}}\,A(\,v,s) - \frac{1}{v(vs)^{\frac{1}{2}-1}}\left[{}_0D_t^{\frac{1}{2}-1}\; f(t)\right]_{t=0}\right) + a\,A\,(v,s) = 0$$

Step3: S implifying the Exponents and initial condition. Note that $\left(\frac{1}{2} - 1 = -\frac{1}{2}\right)$. The term involing the initial condition becanes.

$$\frac{1}{v(vs)^{-\frac{1}{2}}} = \frac{(vs)^{\frac{1}{2}}}{v} = \frac{\sqrt{v}\,\sqrt{s}}{v} = \frac{\sqrt{s}}{\sqrt{v}}$$

substitute the given initial value

$$\left[{}_0D_t^{-\frac{1}{2}}\; f(t)\right]_{t=0} = c$$

$$\rightarrow \frac{1}{\sqrt{\mathrm{vs}}}\,\mathrm{A(v,s)} - \frac{\sqrt{\mathrm{s}}}{\sqrt{\mathrm{v}}}\,\mathrm{c} + \mathrm{a}\,\mathrm{A(v,s)} = 0$$

Step4: Algebraic isolation of $\mathrm{A(v,s)}$ Multiply the entire equation by $\sqrt{vs}$ to clear the denominators:

$$A(v,s) - \left[\frac{\sqrt{s}}{\sqrt{v}} - \sqrt{v}\,\sqrt{s}\right]c \,+\, a\,\sqrt{vs}\,A(v,s) = 0$$

$$\rightarrow\; A(v,s) - sc + a\,\sqrt{vs}\,A(v,s) = 0$$

factor out $\mathrm{A(v,s)}$:

$$\mathrm{A(v,s)}\left[1 + a\sqrt{vs}\right] = SC$$

Thus, the solution in the transform domain is

$$\mathrm{A(v,s)} = \frac{C\,S}{1 + a\sqrt{vs}\;^{\frac{1}{2}}}$$

Inverse transform and final solution to find (t) , we apply the inverse Abaoub - Shikheam transform. Based on the transform tables for fractional orders, the form $\frac{s}{1+\,ks^{\alpha}}$ corresponds to the Mittag - Leffler function.

$$f(t) = ct^{-\frac{1}{2}}\,E_{\frac{1}{2},\frac{1}{2}}\!\left(\,-a\sqrt{t}\right)$$

Where the Millag - Leffler function $E_{\propto,\beta}\,(z)$ is efined as:

$$E_{\frac{1}{2},\frac{1}{2}}(Z) = \sum_{k=0}^{\infty} \frac{Z^k}{\Gamma(\frac{k}{2} + \frac{1}{2})}$$

4.2. The Mellin Transform method:

The Mellin transform $M\{f(t); Z\} = F(Z)$ maps the differential operator into a shift operator in the complex plane. The transform of the Rimann- Liqoulille derivative given by:

$$M\{{}_0D_t^{\propto}\, f(t); z\} = \frac{\Gamma(1 - Z + \propto)}{\Gamma(1 - Z)}\,F(Z - \propto) \quad \rightarrow \text{①}$$

Step1: Applying the trans form apply Mellin Transform method to both sides of sides of Eq:

$$M\left[{}_0D_t^{\frac{1}{2}}\, f(t)\right] + a\,M[f(t)] = 0 \quad \rightarrow \text{②}$$

$$\Rightarrow \frac{\Gamma(1 - Z + \propto)}{\Gamma(1 - Z)}\;F(Z - \propto) + a\,F\,(Z) = 0$$

$$\Rightarrow \frac{\Gamma(1 - Z + {}^{1}/_{2})}{\Gamma(1 - Z)}\;F\left(Z - \frac{1}{2}\right) + a\,F\,(Z) = 0$$

$$\Rightarrow \frac{\Gamma({}^{3}/_{2} - Z)}{\Gamma(1 - Z)}\;F\left(Z - \frac{1}{2}\right) + a\,F\,(Z) = 0$$

Step2: Solve for $\mathrm{F}(s)$ via iteration rearrange as an ecurrence relation**:**

$$F(Z) = -\,\frac{1}{a}\,,\frac{\Gamma({}^{3}/_{2} - Z)}{\Gamma(1 - Z)}\quad F\left(Z - {}^{1}/_{2}\right)$$

By iterative reduction to satisfy the initial condition **C**, we chlain the general solution in the Mellin domain:

$$F(Z) = \;c.\;\; a^{-(2Z-1)}\;\frac{\Gamma(Z)\,\Gamma({}^{1}/_{2} - z)}{\Gamma({}^{1}/_{2})}$$

Step4: Complex inversion. Mettin-Barnes integral apply the inverse transform formula:

$$F(t) = \frac{c}{a\sqrt{\pi}} \cdot \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} (a^2 t)^{-z} \lceil Z({}^{1}/_{2} - Z)\, dZ$$

step5 : Apply the Residue Theorm Evaluate the residues at the simple Poles of $\lceil(Z)$ located at $Z = -k$ **for** $k = 0, 1, 2, \ldots$

$$F(t) = \frac{c}{a\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^K}{K1} \left\lceil \left(k \frac{1}{2}\right) (a^2 t)^k \, a^{-1} \right.$$

Using the duplication formula for Gamma function this simplifies to:

$$F(t) = c\, t^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\left(-a\sqrt{t}\right)^K}{\left\lceil \left(\frac{K+1}{2}\right) \right.}$$

Final exact soulation.

The resulting Power series is the definition of the Mittag-Function.

$$F(t) = c\, t^{-\frac{1}{2}} \, E_{\frac{1}{2},\frac{1}{2}}\left(-a\sqrt{t}\right)$$

Example:

Find the solution to the following equations using Laplaces Frans form Methed **?**

$${}_0D_t^{\frac{1}{2}} \;\; f(t) + \, a\, f(t) = 0, \qquad t > 0, \left[{}_0D_t^{-\frac{1}{2}} f(t)\right]_{t=0} = c \to \text{①}$$

Take the Laplace transform for both sides using the law:

$$\mathcal{L}\left[{}_0D_t^{\frac{1}{2}} f(t)\right] + \, a\, \mathcal{L}\, f(t) = \, 0 \longrightarrow \text{②}$$

Note:

$$\int_0^{\infty} e^{-st} \; {}_0D_t^{\propto} f(t)\, dt = \, S^{\propto} \;\, F(s) - \sum_{k=0}^{n-1} S^k \left[{}_0D_t^{\propto-k-1} \;\, f(t)\right]_{t=0}$$

$$\Longrightarrow \, s^{\propto} f(s) - \sum_{k=0}^{n-1} S^k \, \left[{}_0D_t^{\propto-\frac{1}{2}-1} \;\; f(t)\right]_{t=0} + a \;\, F(s) = 0 \to \text{③}$$

$$\Longrightarrow \, s^{\frac{1}{2}} F(s) - \sum_{k=0}^{n-1} S^k \, \left[{}_0D_t^{\propto-\frac{1}{2}-1} \;\; f(t)\right]_{t=0} + aF(s) = 0$$

$$k = 0, \to > n - 1 = 0 \to n = 0$$

$$\Longrightarrow \, s^{\frac{1}{2}} F(s) - \, \left[{}_0D_t^{\propto-\frac{1}{2}-1} \;\; f(t)\right]_{t=0} + a\, F(s) = 0$$

Compensation based on the initial condition

$$\Longrightarrow \, s^{\frac{1}{2}} F(s) - c + a \;\, F(s) = 0 \;\to \text{④}$$

Takr acomman factor, $F(s)$:

$$\Longrightarrow F(s) \, \left[s^{\frac{1}{2}} + a\right] = 0 \;\to \text{⑤}$$

$$\Longrightarrow F(s) = \, \frac{c}{\left(s^{\frac{1}{2}} + a\right)}$$

Finding the inverse laplace transform using Mittag- leffler function.

$$\mathcal{L}^{-1} \left[\frac{1}{(s^{\propto} + a)}\right] = \, t^{\propto-1} \, E_{\propto,\propto} \, (-at^{\propto})$$

$$s.t \;\; \propto = \frac{1}{2}$$

$$\mathcal{L}^{-1} \, (f(t)) = \mathcal{L}^{-1} \left[\frac{1}{\left(s^{\frac{1}{2}} + a\right)}\right],$$

$$\Longrightarrow f(t) = C.\left[t^{\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(-at^{\frac{1}{2}})\right]$$

$$\Longrightarrow f(t) = C.\left[t^{\frac{1}{2}-1} E_{\frac{1}{2},\frac{1}{2}} \left(-at^{\frac{1}{2}}\right)\right] \;\to \text{⑥}$$

Using the series expansion

$$E_{\alpha,\beta}^{(z)} = \sum_{k=0}^{\infty} \frac{z^k}{\lceil \alpha\, k + \beta}, \qquad \alpha > 0 \;, \qquad \beta > 0$$

it is easy to check that for $\propto = \mathbf{1}$ the solution **(1.16)** is identical the solution.

$$f(t) = C\left[\frac{1}{\sqrt{\pi t}} - e^{t}\, erfc(\sqrt{t})\right].$$

5. Final synthesis and comparative analysis.

Table 1: Comparative Analysis of Solution Methods for Fractional Differential Equations

Derivation Mechanism	**Achieved Result**	**Research Value Addition**	**Methodology**
Direct Algebraic Integration	$f(t) = ct^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(-a\sqrt{t})$	Computational efficiency and modeling	Laplace Transform
Complex Residue a Pole Analysis	$f(t) = ct^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(-a\sqrt{t})$	Analytic depth and asymptotic behavior insight	Mellin Transform
Dual-Param-Etric scaling	$f(t) = ct^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(-a\sqrt{t})$	Modern Flexibility In Physical Scaling	Abaoub-shkhean

Conclusion

This research provides a rigorous comparative analysis of the Laplace Mettin, and Abaoub-shkheam transform in solving fractional differential equations.

The findings demonstrate that while all three methodologies converge to the exact same analytical solution involving the Mittag – Leffler function, they offer different mathematical

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