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A structural and comparative analysis of matrix Riccati and Lyapunov equations in closed-loop optimal control systems

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تحليل بنيوي ومقارن لمعادلتي ريكاتي و ليابونوف المصفوفتين في أنظمة التحكم الأمثل ذات الحلقة المغلقة

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Abstract:
This paper presents a comparative, conceptual, and non computational review of the continuous time matrix Riccati and Lyapunov equations and it focuses on their structural properties, solvability conditions, and theoretical integration within linear dynamical systems. The paper illustrates the explicit transition from the nonlinear quadratic structure of the algebraic Riccati equation to the linear operator of the matrix Lyapunov equation, while also examining the existence of stable, positive semi definite solutions. the paper provides a conceptual analysis of the mechanism by which the Lyapunov operator verifies the stability limits of feedback gains for a control system designed via the Riccati equation, ensuring the system is secure, stable, and resilient against external conditions.

Keywords: Matrix equations, Riccati equation, Lyapunov stability, continuous time systems, optimal control, linear operators.

الملخص
تقدم هذه الورقة مراجعة مقارنة ومفاهيمية -غير قائمة على الحسابات الرقمية- لمعادلتي ريكاتي وليابونوف المصفوفتين في الزمن المستمر، مع التركيز على خصائصهما الهيكلية، وشروط قابليتهما للحل، وتكاملهما النظري ضمن النظم الديناميكية الخطية. وتوضح الورقة الانتقال الصريح من البنية التربيعية غير الخطية لمعادلة ريكاتي الجبرية إلى المؤثر الخطي لمعادلة ليابونوف المصفوفية، كما تبحث في وجود حلول مستقرة وشبه محددة موجبة وتقدم الورقة تحليلاً مفاهيمياً للآلية التي يتحقق من خلالها مؤثر ليابونوف من حدود استقرار مكاسب التغذية الراجعة في نظم التحكم المصممة باستخدام معادلة ريكاتي، مما يضمن كون النظام آمناً ومستقراً وقادراً على الصمود أمام الظروف الخارجية.

الكلمات المفتاحية: المعادلات المصفوفيّة، معادلة ريكاتي، استقرار ليابونوف، الأنظمة ذات الزمن المستمر، التحكم الأمثل، المؤثرات الخطية.

Introduction

The study of matrix equations is a fundamental pillar in the analysis of modern optimal control systems and complex systems such as aircraft, industrial plants, and robots. Among the most prominent matrix formulations used for designing feedback parameters, guiding systems, and assessing system stability [1] ,are the Riccati and Lyapunov equations. The Riccati equation originated in the 18th century as a scalar differential equation

introduced by the Italian mathematician Jacopo Riccati; it was subsequently reformulated into a matrix equation in 1960 by Kalman and has since been applied in Linear Quadratic Regulator control (LQR) [2], signal filtering, control theory, and differential game theory.

The Russian scientist Aleksandr Lyapunov introduced his equation in 1892, and it underwent further development in the mid 20th century. It is used to test the stability of dynamic systems and to design and analyze linear and nonlinear control systems, without the need to solve complex differential equations [3] .

This paper presents a qualitative review of the continuous-time algebraic Riccati and Lyapunov equations, focusing on their properties and the conditions for the existence of solutions, without employing software simulations. The analysis centers on invariant subspaces and operator mappings that integrate both equations of mathematical systems theory[4].

-The matrix Riccati approach: structural properties and solvability:

The continuous-time matrix Riccati equation appears in the Linear Quadratic Regulator (LQR), and its algebraic form is formulated as follows:

$$A^T X + XA - XBR^{-1}B^T X + Q = 0 \qquad (1)$$

where $X \in R^{n\times n}$ it is the unknown symmetric solution matrix,

$A \in R^{n\times n}$ is the state matrix, $B \in R^{n\times m} is\ the$ input matrix, $Q \in R^{n\times n}$ is the weight of the case , $R \in R^{m\times m}$ is the positive definite control weight.

The presence of the nonlinear quadratic term $XBR^{-1}B^T X$ hinders the application of linear superposition principles to find solutions.

The existence of a unique, stable, positive semi definite solution $X \geq 0$ depends on the intrinsic properties of the matrices. According to modern control theory, a stable solution exists if the matrix pairs A, B and A, Q are stabilizable and detectable, respectively; these mathematical properties guarantee the existence of the solution [5,6].

- The continuous time Lyapunov operator: Stability and uniqueness

Unlike the Riccati equation, the continuous-time Lyapunov equation is a linear matrix operator. Given a quadratic Lyapunov function $V(x) = X^T PX$ and its derivative along the system trajectories, and upon analyzing the autonomous linear system, one arrives at the continuous time algebraic Lyapunov equation (CALE)

$$A^T P + PA = -Q \qquad (2)$$

where $P \in R^{n\times n}$ is an unknown symmetric matrix, $Q \in R^{n\times n}$ is a symmetric positive definite matrix.

In Lyapunov stability theory, the state matrix A is asymptotically stable if, for every positive-definite matrix Q, there exists a unique, symmetric, positive definite solution $p > 0$ this property makes the operator a valuable tool meaning it neither generates control signals nor alters the system matrices serving instead solely as a measure

for verifying stability[7,8].

- Theoretical coupling and The Algebraic convergence via closed loop mappings:

Although the algebraic Riccati equation is linear in structure, its stability condition is linked to a linear Lyapunov operator.

The intersection of the two equations emerges from the evaluation of the stable symmetric solution by defining the closed loop state feedback matrix as follows:

$$A_c = A - BR^{-1}B^T X \qquad (3)$$

By substituting A from Equation (3) which is the open loop state matrix into Equation (1) , we obtain the following:

$$(A_c + BR^{-1}B^T X)^T X + X(A_c + BR^{-1}B^T X) - XBR^{-1}B^T X + Q = 0$$

After removing the transformation and eliminating the non linear terms to facilitate the derivation of the Lyapunov equation, we obtain the following equation:

$$A_c^T + XA_c + XBR^{-1}B^T X + Q = 0 \qquad (4)$$

Assuming that $Q_c = XBR^{-1}B^T X + Q$ and substituting this into Equation (4) , the equation becomes

$$A_c^T + XA_c + Q_c = 0 \qquad (5)$$

which is the standard Lyapunov equation.

This means we do not need to prove the stability of the Riccati system from scratch; simply transforming it into a Lyapunov equation automatically guarantees stability[9,10].

-Comparative evaluation: invariant subspace and integral representations:

In the preceding section, the two equations intersected algebraically at a specific point, a phenomenon known as a closed loop which resulted in a geometric discrepancy between them when evaluating the solution spaces [11].

Linearity in the Lyapunov matrix allowed for an analytical solution via direct integration.

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt \qquad (6)$$

The presence of the nonlinear quadratic term in the Riccati equation precludes an integral solution; to circumvent this issue, we construct an augmented linear Hamiltonian matrix of order $2n \times 2n$ which takes the following form[12]:

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \qquad (7)$$

The steady state matrix will guarantee a single, positive solution for the Lyapunov operator, whereas the Riccati equation has multiple solutions[13].

Since we are dealing with a set of solutions, we must isolate the stable invariant subspace as we seek only a single stable solution which we find at the matrix's negative eigenvalues [14].

The Lyapunov equation serves as a passive tool for assessing system stability, whereas the Riccati equation acts as an active tool for designing and determining optimal control parameters[15]

Conclusion

This paper presents a comparative analysis independent of numerical computations of the continuous-time matrix Riccati and Lyapunov equations.

It demonstrates that upon closing the control loop and applying feedback gains, the nonlinear algebraic Riccati equation an active tool for control design transforms into the closed loop Lyapunov equation; the latter, characterized by linearity, serves as a passive tool for stability analysis. Thus, algebraic properties constitute the fundamental basis for mathematical proofs in control theory.

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